

PARISIAN QUASI-STATIONARY DISTRIBUTIONS FOR ASYMMETRIC LÉVY PROCESSES

IRMINA CZARNA AND ZBIGNIEW PALMOWSKI

ABSTRACT. In recent years there has been some focus on quasi-stationary behaviour of an one-dimensional Lévy process, where we ask for the law $\mathbb{P}(X_t \in dy | \tau_0^- > t)$ for $t \rightarrow \infty$ and $\tau_0^- = \inf\{t \geq 0 : X_t < 0\}$. In this paper we address the same question for so-called Parisian ruin time τ^θ , that happens when process stays below zero longer than independent exponential random variable with intensity θ .

KEYWORDS. quasi-stationary distribution, Lévy process, ruin probability, asymptotics, Parisian ruin, risk process

CONTENTS

1. Introduction	2
2. Preliminaries	3
3. Main results	6
3.1. Spectrally positive case	6
3.2. Spectrally negative case	10
Appendix	13
References	17

Date: April 15, 2014.

2000 *Mathematics Subject Classification.* 60J99, 93E20, 60G51.

This work is partially supported by the Ministry of Science and Higher Education of Poland under the grants DEC-2013/09/B/HS4/01496 (2014-2016). All the authors kindly acknowledge partial support by the project RARE -318984, a Marie Curie IRSES Fellowship within the 7th European Community Framework Programme.

1. INTRODUCTION

Let $X = \{X_t : t \geq 0\}$ be a spectrally one-sided Lévy process defined on the filtered space $(\Omega, \mathcal{F}, \mathbb{F}, P)$ where the filtration $\mathbb{F} = \{\mathcal{F}_t : t \geq 0\}$ is assumed to satisfy the usual conditions for right continuity and completion. Suppose now that probabilities $\{\mathbb{P}_x\}_{x \in \mathbb{R}}$ corresponds to the conditional version of \mathbb{P} where $X_0 = x$ is given. We simply write $\mathbb{P}_0 = \mathbb{P}$. We assume that $X_t \rightarrow -\infty$ a.s. as $t \rightarrow \infty$.

Define the first passage time into the lower half line $(-\infty, 0)$ by

$$\tau_0^- = \inf\{t \geq 0 : X_t < 0\}.$$

In recent years there has been some focus on the existence and characterization of the so-called limiting quasi-stationary distribution (or Yaglom's limit) defined by:

$$(1) \quad \mu(B) := \lim_{t \uparrow \infty} P_x(X_t \in B | \tau_0^- > t)$$

for $B \in \mathcal{B}([0, \infty))$. The sense in which this limit is quasi-stationary follows the classical interpretations of works such as Seneta and Vere-Jones [18], Tweedie [19], Iglehart [10] (for a random walk), Jacka and Roberts [11], Kyprianou [13] (within the context of the $M/G/1$ queue), Martinez and San Martin [17] (for a Brownian motion with drift), Kyprianou and Palmowski [15] (for a general light-tailed Lévy process), Hass and Rivero [8] (for regularly varying Lévy process), Mandjes et al. [16] (for a workload process of single server queue) and other references therein.

In this short paper, the principal object of interest is the quasi-stationary distribution of Parisian type, where τ_0^- is replaced by so-called Parisian ruin time:

$$(2) \quad \tau^\theta = \inf\{t > 0 : t - \sup\{s < t : X_s \geq 0\} \geq e_\theta, X_t < 0\},$$

where e_θ is independent of X exponential random variable with intensity θ . The ruin time τ^θ happens when process X_t stays negative longer than e_θ . We want to emphasize that in the definition of τ^θ there is not a single underlying exponential random variable but a whole sequence (each one of them attached to a separate excursion below zero). The name for this ruin comes from Parisian option that prices are activated or canceled depending on type of option if underlying asset stays above or below barrier long enough in a row (see [2, 4, 6]). So far only probability of Parisian ruin is known (see [5] and [3]). In this paper we will find sufficient conditions for existence and identify via its Laplace transform the following limit:

$$(3) \quad \lim_{t \uparrow \infty} P_x(X_t \in B | \tau^\theta > t) = \mu_x^\theta(B), \quad B \in \mathcal{B}([0, \infty)).$$

The idea of the proof of the main results is based on finding double Laplace transform of $\mathbb{P}_x(X_t \in dy, \tau^\theta > t)$ with respect to space and time. Then for some specific form of the Lévy measure (that will be defined later) using heavy-side operation we will identify the asymptotics of this probability as $t \rightarrow \infty$ (see e.g. [1] and [9]).

The paper is organized as follows. In the next section we state some preliminary facts. Later we give the main result. Finally, in the Appendix we give the proofs of technical lemmas.

2. PRELIMINARIES

In this section we give all results for a spectrally negative Lévy process $X = \{X_t\}_{t \geq 0}$, that is a Lévy process with the Lévy measure Π_X satisfying $\Pi_X(0, \infty) = 0$. With X we associate the Laplace exponent $\varphi(\beta) := \frac{1}{t} \log \mathbb{E}(e^{\beta X_t})$ defined for all $\beta \geq 0$ and function $\Phi(q) = \sup\{\beta \geq 0 : \varphi(\beta) = q\}$. We will consider also dual process $\hat{X}_t = -X_t$ which is a spectrally positive Lévy process with the Lévy measure $\Pi_{\hat{X}}(0, y) = \Pi_X(-y, 0)$. Characteristics of \hat{X} will be indicated by using a hat over the existing notation for characteristics of X . In particular, the probabilities $\hat{\mathbb{P}}_x$ and the expectations $\hat{\mathbb{E}}_x$ concern the dual process.

For the process X we define the ascending ladder height process $(L^{-1}, H) = \{(L_t^{-1}, H_t)\}_{t \geq 0}$:

$$L_t^{-1} := \begin{cases} \inf\{s > 0 : L_s > t\} & \text{if } t < L_\infty \\ \infty & \text{otherwise} \end{cases}$$

and

$$H_t := \begin{cases} X_{L_t^{-1}} & \text{if } t < L_\infty \\ \infty & \text{otherwise,} \end{cases}$$

where $L = \{L_t\}_{t \geq 0}$ is a local time at the maximum (see [14, p. 140]). Recall that (L_t^{-1}, H_t) is a (killed) bivariate subordinator with the Laplace exponent $\kappa(\alpha, \beta) = -\frac{1}{t} \log \mathbb{E}(e^{-\alpha L_t^{-1} - \beta H_t} \mathbf{1}_{\{t \leq L_\infty\}})$ and with the jump measure Π_H . We define the descending ladder height process $(\hat{L}^{-1}, \hat{H}) = \{(\hat{L}_t^{-1}, \hat{H}_t)\}_{t \geq 0}$ with the Laplace exponent $\hat{\kappa}(\alpha, \beta)$ constructed from dual process \hat{X} . Moreover, from the Wiener-Hopf factorization we have:

$$(4) \quad \kappa(\alpha, \beta) = \Phi(\alpha) + \beta, \quad \hat{\kappa}(\alpha, \beta) = \frac{\alpha - \varphi(\beta)}{\Phi(\alpha) - \beta};$$

see [14, p. 169-170].

We introduce the renewal function:

$$V(dx) = \int_{[0, \infty)} \mathcal{U}(dx, ds) = \mathbb{E} \left(\int_0^\infty \mathbf{1}_{\{H_t \in dx\}} dt \right).$$

For spectrally negative Lévy process upward ladder height process is a linear drift and hence the renewal measure is just Lebesgue measure:

$$(5) \quad V(dx) = dx.$$

Moreover, from [14, p. 195]:

$$(6) \quad \int_0^\infty e^{-\alpha z} \widehat{V}(dz) = \frac{\alpha}{\varphi(\alpha)}.$$

We will also use the first passage times:

$$\tau_x^- = \inf\{t \geq 0 : X_t \leq x\}, \quad \tau_x^+ = \inf\{t \geq 0 : X_t > x\}.$$

We define Girsanov-type change of measure via:

$$(7) \quad \frac{d\mathbb{P}_x^c}{d\mathbb{P}_x} \Big|_{\mathcal{F}_t} = \frac{\mathcal{E}_t(c)}{\mathcal{E}_0(c)}$$

for any c for which $\mathbb{E}e^{cX_1} < \infty$, where $\mathcal{E}_t(c) = \exp\{cX_t - \varphi(c)t\}$ is exponential martingale under \mathbb{P}_x and \mathcal{F}_t is a natural filtration of X . It is easy to check that under this new measure, X still remains within the class of spectrally negative processes. All quantities calculated for \mathbb{P}^c will have subindex c added to their counterparts defined on \mathbb{P} .

For $q \geq 0$, there exists a function $W^{(q)} : [0, \infty) \rightarrow [0, \infty)$, called *q-scale function*, that is continuous and increasing with the Laplace transform:

$$(8) \quad \int_0^\infty e^{-\alpha x} W^{(q)}(x) dx = (\varphi(\alpha) - q)^{-1}, \quad \alpha > \Phi(q).$$

We denote $W^{(0)}(x) = W(x)$. Domain of $W^{(q)}$ is extended to the entire real axis by setting $W^{(q)}(x) = 0$ for $x < 0$. For each $x \geq 0$, function $q \rightarrow W^{(q)}(x)$ may be analytically extended to $q \in \mathcal{C}$. Moreover, let

$$Z^{(q)}(x) = 1 + q \int_0^x W^{(q)}(y) dy.$$

It is known that:

$$(9) \quad \mathbb{E}_x \left(e^{-\alpha \tau_y^+}, \tau_y^+ < \infty \right) = e^{-\Phi(\alpha)(y-x)},$$

$$(10) \quad E_x \left(e^{-q \tau_0^-}, \tau_0^- < \infty \right) = Z^{(q)}(x) - \frac{q}{\Phi(q)} W^{(q)}(x),$$

$$(11) \quad \mathbb{E}_x \left(e^{-q \tau_0^- + v X_{\tau_0^-}}, \tau_0^- < \infty \right) = e^{vx} \left(Z_v^{(p)}(x) - \frac{p}{\Phi_v(p)} W_v^{(p)}(x) \right),$$

where $p = q - \varphi(v)$ and

$$(12) \quad W_v^{(p)}(x) = e^{-vx} W^{(q)}(x),$$

$$(13) \quad Z_v^{(p)}(x) = 1 + p \int_0^x W_v^{(p)}(y) dy = 1 + p \int_0^x e^{-vy} W^{(q)}(y) dy.$$

We introduce the notation $Z^{(q),\beta}(x) = 1 + (q - \varphi(\beta)) \int_0^x e^{-\beta y} W^{(q)}(y) dy$. Note that $Z^{(q),0}(x) = Z^{(q)}(x)$.

Further, for $p_2 = q - \widehat{\varphi}(\alpha) = q - \varphi(-\alpha)$ and $p_3 = p_2 + \theta$ we have

$$\widehat{Z}_\alpha^{(p_2)}(x) = \widehat{Z}^{(q),\alpha}(x),$$

$$\widehat{W}_\alpha^{(p_2)}(x) = e^{-\alpha x} \widehat{W}^{(q)}(x),$$

$$\widehat{Z}_\alpha^{(p_3)}(x) = \widehat{Z}^{(q+\theta),\alpha}(x),$$

$$\widehat{W}_\alpha^{(p_3)}(x) = e^{-\alpha x} \widehat{W}^{(q+\theta)}(x).$$

Assume from now on that:

(A1): the Laplace exponent φ is well-defined for $0 < \theta < \theta_r$, where $\theta_r > 0$,

(A2): $\varphi(\theta)$ attains its negative infimum ξ^* at $0 < Q^* \leq \theta_r$ with $\varphi'(Q^*) = 0$,

(A3): X_1 is non-lattice.

In [15] and [16] it was proved that the quasi-stationary distribution (1) related with the classical ruin time τ_0^- equals

$$\mu(dy) = \xi^* \kappa_{\xi^*}(0, \xi^*) e^{-\xi^* y} V_{\xi^*}(y) dy \quad \text{on } [0, \infty).$$

In particular if X is a spectrally negative processes, then

$$\mu(dy) = (\xi^*)^2 y e^{-\xi^* y} dy$$

and if X is spectrally positive, then

$$\mu(dy) = (\xi^*)^2 e^{-\xi^* y} V_{\xi^*}(y) dy$$

with the Laplace transform $\int_{[0,\infty)} e^{-\alpha y} \mu(dy) = \frac{-\varphi(\xi^*)}{\varphi(-\alpha) - \varphi(\xi^*)}$. The key lemma in deriving above results comes from Wiener-Hopf factorization (see [15, eq. (10)]) which holds true for any Lévy process:

Lemma 1. *We have:*

$$\mathbb{E}_x [e^{-\alpha X_{e_q}}, \tau_0^- > e_q] = \frac{\kappa(q, 0)}{\kappa(q, \alpha)} e^{-\alpha x} \left(\int_0^x e^{\alpha z} \mathbb{P} \left(\widehat{X}(e_q) \in dz \right) \right),$$

where $\overline{X}(t) = \sup_{s \leq t} X(s)$.

To obtain main result we will specialize theorem [7, Th. 37.1] in the following way. We first recall a concept of \mathfrak{W} -contour with an half-angle of opening $\phi/2 < \phi \leq \pi$ as it is in [7, Fig 30, p. 240] and $\mathcal{G}_\alpha(\phi)$ is the region between contour \mathfrak{W} and line $\Re(z) = 0$.

Proposition 2. *Suppose that $\tilde{f}(z) = \int_0^\infty e^{-zx} f(x) dx$ (defined for some function f) satisfies the following three conditions for some $\alpha < 0$:*

- (A1): $\tilde{f}(z)$ is analytic in the region $\mathcal{G}_\alpha(\phi)$,
- (A2): $\tilde{f}(z) \rightarrow 0$ as $|z| \rightarrow \infty$ for $z \in \mathcal{G}_\alpha(\phi)$,
- (A3): for some constant K and non integer real number s

$$\tilde{f}(z) = K \mathbb{1}(s > 0) - C(z - \alpha)^s + o((z - \alpha)^s),$$

for $\mathcal{G}_\alpha(\phi) \ni z \rightarrow \alpha$.

Then

$$f(x) = \frac{C}{\Gamma(-s)} x^{-s-1} e^{\alpha x} (1 + o(1)),$$

where K must be $\tilde{f}(\alpha)$ if $s > 0$ and recall that some function $h(x) = o(1)$ if $\lim_{x \rightarrow \infty} h(x) = 0$.

3. MAIN RESULTS

3.1. Spectrally positive case. In this section we assume that X is a spectrally positive Lévy process (bounded or with $\sigma > 0$) satisfying conditions (A1)-(A3). Moreover, we will also assume the following condition.

(A4): Function $\widehat{\Phi}$ can be analytically extended into $\mathcal{G}_{\xi^*}(\psi)$ for some $\pi/2 < \psi \leq \pi$.

Remark 3. Since $\widehat{\Phi}(\vartheta)$ is the Laplace exponent of a subordinator we have the following spectral representation:

$$(14) \quad \widehat{\Phi}(\vartheta) = d_+ \vartheta + \int_0^\infty (1 - e^{-\vartheta y}) \Pi_+(dy),$$

and $\int_0^\infty (y \wedge 1) \Pi_+(dy) < \infty$. From its definition we see that ξ^* must be a singular point of $\widehat{\Phi}$. Moreover, if there exists a density of Π_+ which is of semiexponential type, then condition (A4) is satisfied. In particular, this assumption holds for example for a linear Brownian motion $X(t) = \sigma B(t) - ct$, where $c > 0$ (see for details [16]).

To obtain main formula we will use such expansion for the Laplace exponent $\widehat{\Phi}(q)$ (for details see [16, Lem. 10]):

$$(15) \quad \widehat{\Phi}(q) = q^* + k^*(q - \xi^*)^{1/2} + o((q - \xi^*)^{1/2}),$$

where $\widehat{\varphi}(q) = \varphi(-q)$ attains its strictly negative minimum at $q^* = -Q^* < 0$, $\xi^* = \widehat{\varphi}(q^*) < 0$, $k^* = \sqrt{2/\widehat{\varphi}''(q^*)}$ and $q \downarrow \xi^*$.

The Laplace transform $\lim_{t \rightarrow \infty} \mathbb{E}[e^{-\alpha X_t} | \tau_0^- > t]$ will be given in terms of function $H_p(\alpha, x)$ defined below:

$$(16) \quad \begin{aligned} H_p(\alpha, x) = & \frac{\xi^*}{(\xi^* - \alpha)^3} \left[-2\xi^* k^* (-e^{-\xi^* x} + e^{-\alpha x}) + k^*(\xi^* - \alpha)(-e^{-\xi^* x} + e^{-\alpha x} + e^{-\xi^* x} \xi^* x) \right. \\ & + \frac{e^{-\xi^* x}}{\theta} \left[(\widehat{\Phi}(\xi^* + \theta) - \xi^*) \xi^* (\xi^* - \alpha) \left(\frac{k^* \theta}{(\widehat{\Phi}(\xi^* + \theta) - \xi^*)^2} - \frac{k^* \xi^* (\xi^* - \varphi(-\alpha))}{(\xi^* - \alpha)^2 (\xi^* + \theta)} \right) \right. \\ & \left. - k^* \left(\frac{\theta}{\widehat{\Phi}(\xi^* + \theta) - \xi^*} + \frac{\xi^* (\xi^* - \varphi(-\alpha))}{(\xi^* - \alpha)(\xi^* + \theta)} - \frac{\xi^* - \varphi(-\alpha) + \theta}{\widehat{\Phi}(\xi^* + \theta) - \alpha} \right) \right. \\ & \left. \left. \left(-2\xi^* \alpha + \widehat{\Phi}(\xi^* + \theta)(\xi^* + \alpha) + (\widehat{\Phi}(\xi^* + \theta) - \xi^*) \xi^* x (\xi^* - \alpha) \right) \right] \right] \end{aligned}$$

when process X is of bounded variation and

$$\begin{aligned}
H_p(\alpha, x) = & \frac{k^* \xi^*}{(\xi^* - \alpha)^4} \left[-2(-e^{-\xi^* x} + e^{-x\alpha})\xi^*(\xi^* - \alpha) \right. \\
& - \frac{e^{-\xi^* x} \xi^* (-\widehat{\Phi}(\xi^* + \theta) + \xi^*)(\widehat{W}'(0)(\xi^* - \alpha)^2 + \widehat{W}^{(q^*),'}(0)\xi^*(\xi^* - \varphi(-\alpha)))}{-\widehat{\Phi}(\xi^* + \theta)\widehat{W}'(0)\xi^* + \widehat{W}'(0)\xi^{*,2} - \widehat{W}^{(q^* + \theta),'}(0)\theta(\xi^* + \theta)} \\
& + (\xi^* - \alpha)^2(-e^{-\xi^* x} + e^{-x\alpha} + e^{-\xi^* x} \xi^* x) - \frac{e^{-\xi^* x}(\xi^* - \alpha)(\xi^* + \theta)}{(\widehat{\Phi}(\xi^* + \theta)\widehat{W}'(0)\xi^* - \widehat{W}'(0)\xi^{*,2} + \widehat{W}^{(q^* + \theta),'}(0)\theta(\xi^* + \theta))^2} \\
& \left(\frac{\widehat{W}^{(q^*),'}(0)\xi^*(\xi^* - \varphi(-\alpha))}{(\xi^* - \alpha)(\xi^* + \theta)} + \frac{\widehat{W}'(0)\xi^*(\alpha - \varphi(-\alpha))}{\alpha(\xi^* + \theta)} - \frac{\widehat{W}^{(q^* + \theta),'}(0)(\xi^* - \varphi(-\alpha) + \theta)}{\widehat{\Phi}(\xi^* + \theta) - \alpha} \right) \\
& \left(2\widehat{\Phi}(\xi^* + \theta)^2 \widehat{W}'(0)\xi^* \alpha + 2\xi^* \alpha (\widehat{W}'(0)\xi^{*,2} - \widehat{W}^{(q^* + \theta),'}(0)\theta(\xi^* + \theta)) \right. \\
& + \widehat{\Phi}(\xi^* + \theta)(-4\widehat{W}'(0)\xi^{*,2} \alpha + \widehat{W}^{(q^* + \theta),'}(0)(\xi^* + \alpha)\theta(\xi^* + \theta)) \\
& \left. \left. + (\widehat{\Phi}(\xi^* + \theta) - \xi^*)\xi^* x (\xi^* - \alpha)(\widehat{\Phi}(\xi^* + \theta)\widehat{W}'(0)\xi^* - \widehat{W}'(0)\xi^{*,2} + \widehat{W}^{(q^* + \theta),'}(0)\theta(\xi^* + \theta)) \right) \right] \\
(17)
\end{aligned}$$

otherwise.

The main result of this section is the following theorem.

Theorem 4. *Assume that conditions (A1)-(A4) hold. Then the Parisian quasi-stationary distribution (3) exists and has the Laplace transform $H_p(\alpha, x)/H_p(0, x)$.*

Remark 5. Consider input process of $M|M|1$ queue:

$$(18) \quad X(t) = \sum_{i=1}^{N(t)} \sigma_i - t,$$

where σ_i (for $i = 1, 2, \dots$) are i.i.d. service times that have an exponential distribution with mean $1/\nu$.

The arrival process is a homogeneous Poisson process $N(t)$ with rate λ ; it is assumed that $\varrho := \lambda/\nu < 1$.

In this case we have

$$\widehat{\varphi}(\alpha) = \alpha - \lambda \left(1 - \frac{\nu}{\alpha + \nu} \right) = \alpha - \frac{\lambda\alpha}{\alpha + \nu},$$

yielding $q^* = \sqrt{\lambda\nu} - \nu < 0$, $\xi^* = -(\sqrt{\nu} - \sqrt{\lambda})^2$ and

$$\widehat{\Phi}(q) = \frac{q + \lambda - \nu + \sqrt{(q + \lambda - \nu)^2 + 4\theta\nu}}{2}$$

and it is easy to show that all assumptions (A1)-(A4) are satisfied (see also [16]). We can also compute exact formula for the Laplace transform $H_p(\alpha, x)/H_p(0, x)$.

Above theorem is a consequence of Proposition 2 and the following proposition.

Proposition 6. *We have:*

$$\mathbb{E}_x [e^{-\alpha X_{e_q}}, \tau^\theta > e_q] = C_p(\alpha, x) + H_p(\alpha, x)(q - \xi^*)^{1/2} + o((q - \xi^*)^{1/2})$$

as $q \downarrow \xi^*$ for some C_p and H_p given in (16) or (17) depending on whether the process is bounded or unbounded variation.

The proof of the Proposition 6 is based on the following intermediate lemma.

Lemma 7. *For $\alpha \geq 0$ and $x > 0$ have:*

$$(19) \quad \mathbb{E}_x [e^{-\alpha X_{e_q}}, \tau^\theta > e_q] = \frac{\widehat{\Phi}(q)q}{(\widehat{\Phi}(q) - \alpha)^2} \left(e^{-\alpha x} - e^{-\widehat{\Phi}(q)x} \right) + e^{-\widehat{\Phi}(q)x} \mathbb{E} [e^{-\alpha X_{e_q}}, \tau^\theta > e_q],$$

where

(i): *if process X is of bounded variation then:*

$$(20) \quad \begin{aligned} \mathbb{E} [e^{-\alpha X_{e_q}}, \tau^\theta > e_q] &= \frac{\widehat{\Phi}(q)q(\widehat{\Phi}(q + \theta) - \widehat{\Phi}(q))}{(\widehat{\Phi}(q) - \alpha)^2 \theta} \left\{ \frac{q(q - \varphi(-\alpha))}{(q + \theta)(\widehat{\Phi}(q) - \alpha)} \right. \\ &\quad \left. - \frac{q - \varphi(-\alpha) + \theta}{\widehat{\Phi}(q + \theta) - \alpha} + \frac{\theta}{\widehat{\Phi}(q + \theta) - \widehat{\Phi}(q)} \right\}, \end{aligned}$$

(ii): *if the Gaussian coefficient σ appearing in the Lévy-Khinchine decomposition is strictly positive then:*

$$(21) \quad \begin{aligned} \mathbb{E} [e^{-\alpha X_{e_q}}, \tau^\theta > e_q] &= \frac{\widehat{\Phi}(q)q}{(\widehat{\Phi}(q) - \alpha)^2} \left\{ \frac{q}{(q + \theta)} \widehat{W}'(0) \left(\frac{q}{\widehat{\Phi}(q)} - \frac{\varphi(-\alpha)}{\alpha} \right) \right. \\ &\quad \left. + \frac{q}{q + \theta} \frac{q - \varphi(-\alpha)}{\widehat{\Phi}(q) - \alpha} \widehat{W}^{(q),'}(0) - \frac{q - \varphi(-\alpha) + \theta}{\widehat{\Phi}(q + \theta) - \alpha} \widehat{W}^{(q + \theta),'}(0) \right\} \\ &\quad : \left\{ \frac{q^2}{(q + \theta)\widehat{\Phi}(q)} \widehat{W}'(0) + \frac{\theta}{\widehat{\Phi}(q + \theta) - \widehat{\Phi}(q)} \widehat{W}^{(q + \theta),'}(0) \right\}. \end{aligned}$$

The proof of Lemma 7 is given in Appendix.

Proof of Proposition 6. We use simple facts saying that if functions a and b have expansions $a(q) = a^* + a^\circ(q - \xi^*)^{1/2} + o((q - \xi^*)^{1/2})$ and $b(q) = b^* + b^\circ(q - \xi^*)^{1/2} + o((q - \xi^*)^{1/2})$ then

$$\begin{aligned} a(q)b(q) &= a^*b^* + (a^\circ b^* + b^\circ a^*)(q - \xi^*)^{1/2} + o((q - \xi^*)^{1/2}), \\ \frac{a(q)}{b(q)} &= \frac{a^*}{b^*} + \left(\frac{a^\circ}{b^*} - \frac{b^\circ}{(b^*)^2} a^* \right) (q - \xi^*)^{1/2} + o((q - \xi^*)^{1/2}), \\ e^{a(q)x} &= e^{a^*x} + a^\circ x e^{a^*x} (q - \xi^*)^{1/2} + o((q - \xi^*)^{1/2}). \end{aligned}$$

Applying these identities to the LHS of (19) with (20) or (21) completes the proof. \square

3.2. Spectrally negative case. We assume in section that X is a spectrally negative Lévy process satisfying conditions (A1)-(A3). Moreover, we will also assume the following condition.

(A5): Function Φ can be extended analytically into $\mathcal{G}_{\xi^*}(\psi)$ for some $\pi/2 < \psi \leq \pi$.

Remark 8. Function $\Phi(\vartheta)$ is the Laplace exponent of a subordinator with representation:

$$(22) \quad \Phi(\vartheta) = d_- \vartheta + \int_0^\infty (1 - e^{-\vartheta y}) \Pi_-(dy),$$

and $\int_0^\infty (y \wedge 1) \Pi_-(dy) < \infty$. From its definition we can see that if there exists a density of Π_- which is of semiexponential type, then assumption (A5) is satisfied; see for details [16].

From [16, Lem. 16] we have such expansion for the Laplace exponent $\Phi(q)$:

$$(23) \quad \Phi(q) = q^* + k^*(q - \xi^*)^{1/2} + o((q - \xi^*)^{1/2}),$$

where $\varphi(q)$ attains its strictly negative minimum at $q^* = Q^* > 0$, $\xi^* = \varphi(q^*) < 0$, $k^* = \sqrt{2/\varphi''(q^*)}$ and $q \downarrow \xi^*$.

Denote

$$(24) \quad G_1(\alpha, x, q) = W^{(q)}(0) + W^{(q)}(x) - e^{-\alpha x} W^{(q)}(0) - \alpha e^{-\alpha x} \int_0^x e^{\alpha z} W^{(q)}(z) dz,$$

$$(25) \quad G_2(\alpha, x, q) = \alpha e^{-\alpha x} \int_0^x e^{\alpha z} \int_0^z W^{(q)}(y) dy dz - \int_0^x W^{(q)}(y) dy.$$

Moreover, the Laplace transform $\lim_{t \rightarrow \infty} \mathbb{E}[e^{-\alpha X_t} | \tau_0^- > t]$ will be given in terms of function $H_p(\alpha, x)$ defined below:

$$\begin{aligned}
 H_n(\alpha, x) = & \frac{k^* \xi^*}{(\xi^* - \alpha)(\xi^* + \alpha)^3 \theta (\xi^* + \theta)} \\
 & \left[-\theta \left(-G_2(\alpha, x, q^*) \xi^{*,3} \alpha + \xi^* W^{(q^*)}(x) \alpha^2 + G_2(\alpha, x, q^*) \xi^* \alpha^3 - e^{\xi^* x} \xi^{*,2} x (\xi^* + \alpha)^2 \right. \right. \\
 & - \xi^{*,2} W^{(q^*)}(x) \varphi(\alpha) - \xi^{*,2} W^{(q^*)}(x) \theta - G_2(\alpha, x, q^*) \xi^{*,2} \alpha \theta + W^{(q^*)}(x) \alpha^2 \theta \\
 & + G_2(\alpha, x, q^*) \alpha^3 \theta + G_1(\alpha, x, q^*) (\xi^{*,2} - \alpha^2) (\xi^* + \theta) \Big) \\
 (26) \quad & \left. + e^{\Phi(\theta+q^*)x} Z^{(q^*), \Phi(\theta+q^*)}(x) (\Phi(\xi^* + \theta) + \alpha) (\xi^* \alpha^2 + \alpha^2 \theta - \xi^{*,2} (\varphi(\alpha) + \theta)) \right]
 \end{aligned}$$

if process X is of bounded variation and

$$\begin{aligned}
 H_n(\alpha, x) = & k^* \xi^* \left[-\frac{G_1(\alpha, x, q^*)}{(\xi^* + \alpha)^2} + \frac{G_2(\alpha, x, q^*) \alpha}{(\xi^* + \alpha)^2} + \frac{\xi^{*,2} W^{(q^*)}(x) (-\xi^* + \varphi(\alpha))}{(\xi^* - \alpha)(\xi^* + \alpha)^3 (\xi^* + \theta)} \right. \\
 & - \frac{W^{(q^*),'}(0) (e^{\Phi(\theta+q^*)x} Z^{(q^*), \Phi(\theta+q^*)}(x) (-\Phi(\xi^* + \theta) + \xi^*) + W^{(q^*)}(x) \theta)}{(\xi^* + \alpha)^2 (-\Phi(\xi^* + \theta)^2 + \Phi(\xi^* + \theta) \xi^* + W^{(q^*),'}(0) \theta)} \\
 & + \frac{e^{\Phi(\theta+q^*)x} Z^{(q^*), \Phi(\theta+q^*)}(x) W^{(q^*),'}(0)}{(\xi^* - \alpha)(\xi^* + \alpha)^2 (\xi^* + \theta) (-\Phi(\xi^* + \theta)^2 + \Phi(\xi^* + \theta) \xi^* + W^{(q^*),'}(0) \theta)^2} \\
 & - \frac{\Phi(\xi^* + \theta) W^{(q^*)}(x) \theta (\xi^{*,2} (\xi^* + \alpha)^2 + W^{(q^*),'}(0) (2\xi^{*,3} - \xi^* \alpha^2 - \alpha^2 \theta + \xi^{*,2} (-\varphi(\alpha) + \theta)))}{(\xi^* - \alpha)(\xi^* + \alpha)^2 (\xi^* + \theta) (-\Phi(\xi^* + \theta)^2 + \Phi(\xi^* + \theta) \xi^* + W^{(q^*),'}(0) \theta)^2} \\
 (27) \quad & \left. + \frac{e^{\xi^* x} \xi^{*,2} x}{(\xi^{*,2} - \alpha^2) (\xi^* + \theta)} \right]
 \end{aligned}$$

otherwise.

The main result of this section is the following theorem.

Theorem 9. *Assume that conditions (A1)-(A3) and (A5) hold. Then the Parisian quasi-stationary distribution (3) exists and has the Laplace transform $H_n(\alpha, x)/H_n(0, x)$.*

Remark 10. Consider now the Brownian motion with drift $X_t = \sigma B_t - t$ for $\sigma > 0$. In this case

$$\varphi(\alpha) = -\alpha + \frac{\sigma^2 \alpha^2}{2},$$

so that $q^* = 1/\sigma^2 > 0$, $\xi^* = -1/(2\sigma^2)$ and

$$\Phi(q) = \frac{1}{\sigma^2} \left(1 + \sqrt{1 + 2\sigma^2 q} \right).$$

Note that all assumptions (A1)-(A3) and (A5) are satisfied (see [16]). We can also compute exact formula for the Laplace transform $H_n(\alpha, x)/H_n(0, x)$.

This main theorem follows straightforward from the following result and Proposition 2.

Proposition 11.

$$\mathbb{E}_x [e^{-\alpha X_{e_q}}, \tau^\theta > e_q] = C_n(\alpha, x) + H_n(\alpha, x)(q - \xi^*)^{1/2} + o((q - \xi^*)^{1/2})$$

as $q \downarrow \xi^*$ for some C_n and H_n given in (26) or (27) depending on whether the process is bounded or unbounded variation.

Proof of Proposition 11 follows from straightforward calculations and the following proposition.

Proposition 12. For $\alpha \geq 0$ and $x > 0$ we have:

$$(28) \quad \begin{aligned} \mathbb{E}_x [e^{-\alpha X_{e_q}}, \tau^\theta > e_q] &= \frac{q}{\Phi(q) + \alpha} G_1(\alpha, x) + \frac{\Phi(q)q}{\Phi(q) + \alpha} G_2(\alpha, x) \\ &\quad - \frac{\Phi^2(q)q}{(\Phi^2(q) - \alpha^2)(q + \theta)} \left(e^{-\alpha x} Z^{(q), -\alpha}(x) - \frac{q - \varphi(\alpha)}{\Phi(q) + \alpha} W^{(q)}(x) \right) \\ &\quad + \frac{\Phi^2(q)qe^{\Phi(q)x}}{(\Phi^2(q) - \alpha^2)(q + \theta)} \\ &\quad + \mathbb{E} [e^{-\alpha X_{e_q}}, \tau^\theta > e_q] \left(e^{\Phi(\theta+q)x} Z^{(q), \Phi(\theta+q)}(x) - \frac{\theta}{\Phi(q + \theta) - \Phi(q)} W^{(q)}(x) \right), \end{aligned}$$

where functions $G_1(\alpha, x)$ and $G_2(\alpha, x)$ are defined in (24) and (25).

Moreover

(i): if process X is of bounded variation then:

$$(29) \quad \begin{aligned} \mathbb{E} [e^{-\alpha X_{e_q}}, \tau^\theta > e_q] &= \frac{q(\Phi(\theta + q) - \Phi(q))}{(\Phi(q) + \alpha)\theta} \\ &\quad - \frac{\Phi^2(q)q(\Phi(\theta + q) - \Phi(q))(q - \varphi(\alpha))}{(\Phi^2(q) - \alpha^2)(q + \theta)\theta(\Phi(q) + \alpha)}, \end{aligned}$$

(ii): if the Gaussian coefficient σ appearing in the Lévy-Khinchine decomposition is strictly positive then:

$$(30) \quad \begin{aligned} \mathbb{E} [e^{-\alpha X_{e_q}}, \tau^\theta > e_q] &= \left\{ \frac{q}{\Phi(q) + \alpha} W^{(q), \prime}(0) + \frac{\Phi^3(q)q}{(\Phi^2(q) - \alpha^2)(q + \theta)} \right. \\ &\quad \left. + \frac{\Phi^2(q)q}{(\Phi^2(q) - \alpha^2)(q + \theta)} \left(\alpha + \frac{q - \varphi(\alpha)}{\Phi(q) + \alpha} W^{(q), \prime}(0) \right) \right\} \\ &\quad : \left\{ \Phi(\theta + q) - \frac{\theta}{\Phi(q + \theta) - \Phi(q)} W^{(q), \prime}(0) \right\}. \end{aligned}$$

APPENDIX

Proof of Lemma 7. For $\epsilon \geq 0$ we define the stopping time:

$$\tau_{-\epsilon}^\theta = \inf\{t > e_\theta : X_{t-e_\theta} = -\epsilon, X_t > 0\},$$

which is the first time when an excursion of length greater than e_θ starting when X is at $-\epsilon$ ending at the moment of getting back up over zero has occurred.

Using strong Markov property we have:

$$\begin{aligned} (31) \quad \mathbb{E}_z [e^{-\alpha X_{e_q}}, \tau_{-\epsilon}^\theta > e_q] \\ = \mathbb{E}_{z+\epsilon} [e^{-\alpha X_{e_q}}, \tau_0^- > e_q] + \mathbb{P}_{z+\epsilon}(\tau_0^- < e_q) \mathbb{E}_{-\epsilon} [e^{-\alpha X_{e_q}}, \tau_{-\epsilon}^\theta > e_q]. \end{aligned}$$

From [16, Thm. 4] and (4) it follows that for $x > 0$ we have:

$$\begin{aligned} (32) \quad \mathbb{E}_x [e^{-\alpha X_{e_q}}, \tau_0^- > e_q] \\ = q \int_0^\infty e^{-qt} \mathbb{E}_x [e^{-\alpha X_t}, \tau_0^- > t] \\ = \frac{\kappa(q, 0)}{\kappa(q, \alpha)} e^{-\alpha x} \left(\int_{(0, x]} e^{\alpha z} \mathbb{P}(\widehat{X}(e_q) \in dz) \right) = \frac{\widehat{\Phi}(q)q}{(\widehat{\Phi}(q) - \alpha)^2} (e^{-\alpha x} - e^{-\widehat{\Phi}(q)x}). \end{aligned}$$

Taking $\epsilon = 0$ in (31) completes the proof of (19) in view of the equation (9).

Moreover, to identify $\mathbb{E}_{-\epsilon} [e^{-\alpha X_{e_q}}, \tau_{-\epsilon}^\theta > e_q]$ note that:

$$\begin{aligned} \mathbb{E}_{-\epsilon} [e^{-\alpha X_{e_q}}, \tau_{-\epsilon}^\theta > e_q] \\ = \int_0^\infty \mathbb{P}_{-\epsilon}(\tau_0^+ > e_q, X_{\tau_0^+} \in dz) \mathbb{P}(e_\theta > e_q) \mathbb{E}_z [e^{-\alpha X_{e_q}}, \tau_{-\epsilon}^\theta > e_q] \\ + \int_0^\infty \mathbb{P}_{-\epsilon}(\tau_0^+ \leq \min(e_q, e_\theta), X_{\tau_0^+} \in dz) \mathbb{E}_z [e^{-\alpha X_{e_q}}, \tau_{-\epsilon}^\theta > e_q]. \end{aligned}$$

This together with (31) and (32) give:

$$\begin{aligned}
& \mathbb{E}_{-\epsilon} \left[e^{-\alpha X_{e_q}}, \tau_{-\epsilon}^\theta > e_q \right] \\
&= \frac{\widehat{\Phi}(q)q^2}{(q+\theta)(\widehat{\Phi}(q)-\alpha)^2} \left\{ e^{-\alpha\epsilon} \left(\mathbb{E}_\epsilon \left[e^{\alpha \widehat{X}_{\widehat{\tau}_0^-}}, \widehat{\tau}_0^- < \infty \right] - \mathbb{E}_\epsilon \left[e^{-q\widehat{\tau}_0^- + \alpha \widehat{X}_{\widehat{\tau}_0^-}}, \widehat{\tau}_0^- < \infty \right] \right) \right. \\
&\quad \left. - e^{-\widehat{\Phi}(q)\epsilon} \left(\mathbb{E}_\epsilon \left[e^{\widehat{\Phi}(q)\widehat{X}_{\widehat{\tau}_0^-}}, \widehat{\tau}_0^- < \infty \right] - \mathbb{E}_\epsilon \left[e^{-q\widehat{\tau}_0^- + \widehat{\Phi}(q)\widehat{X}_{\widehat{\tau}_0^-}}, \widehat{\tau}_0^- < \infty \right] \right) \right\} \\
&\quad + \frac{\widehat{\Phi}(q)q}{(\widehat{\Phi}(q)-\alpha)^2} \left\{ e^{-\alpha\epsilon} \mathbb{E}_\epsilon \left[e^{-(q+\theta)\widehat{\tau}_0^- + \alpha \widehat{X}_{\widehat{\tau}_0^-}}, \widehat{\tau}_0^- < \infty \right] \right. \\
&\quad \left. - e^{-\widehat{\Phi}(q)\epsilon} \mathbb{E}_\epsilon \left[e^{-(q+\theta)\widehat{\tau}_0^- + \widehat{\Phi}(q)\widehat{X}_{\widehat{\tau}_0^-}}, \widehat{\tau}_0^- < \infty \right] \right\} \\
&\quad + e^{-\widehat{\Phi}(q)\epsilon} \mathbb{E}_{-\epsilon} \left[e^{-\alpha \widehat{X}_{e_q}}, \widehat{\tau}_{-\epsilon}^\theta > e_q \right] \left\{ \frac{q}{q+\theta} \mathbb{E}_\epsilon \left[e^{\widehat{\Phi}(q)\widehat{X}_{\widehat{\tau}_0^-}}, \widehat{\tau}_0^- < \infty \right] \right. \\
&\quad \left. - \frac{q}{q+\theta} \mathbb{E}_\epsilon \left[e^{-q\widehat{\tau}_0^- + \widehat{\Phi}(q)\widehat{X}_{\widehat{\tau}_0^-}}, \widehat{\tau}_0^- < \infty \right] + \mathbb{E}_\epsilon \left[e^{-(q+\theta)\widehat{\tau}_0^- + \widehat{\Phi}(q)\widehat{X}_{\widehat{\tau}_0^-}}, \widehat{\tau}_0^- < \infty \right] \right\}.
\end{aligned}$$

Then from (10) and (11):

$$\begin{aligned}
& \mathbb{E}_{-\epsilon} \left[e^{-\alpha X_{e_q}}, \tau_{-\epsilon}^\theta > e_q \right] \times \\
& \left(1 - \frac{q}{q+\theta} \left(\widehat{Z}^{(0), \widehat{\Phi}(q)}(\epsilon) - \frac{q}{\widehat{\Phi}(q)} e^{-\widehat{\Phi}(q)\epsilon} \widehat{W}(\epsilon) \right) \right. \\
& \quad \left. + \frac{q}{q+\theta} - \widehat{Z}^{(\theta+q), \widehat{\Phi}(q)}(\epsilon) + \frac{e^{-\widehat{\Phi}(q)\epsilon} \theta}{\widehat{\Phi}(q+\theta) - \widehat{\Phi}(q)} \widehat{W}^{(q+\theta)}(\epsilon) \right) \\
&= \frac{\widehat{\Phi}(q)q}{(\widehat{\Phi}(q)-\alpha)^2} \left\{ \right. \\
& \quad \frac{q}{q+\theta} \left(\widehat{Z}^{(0), \alpha}(\epsilon) - \frac{\varphi(\alpha)}{\alpha} e^{-\alpha\epsilon} \widehat{W}(\epsilon) - \widehat{Z}^{(0), \widehat{\Phi}(q)}(\epsilon) + \frac{q}{\widehat{\Phi}(q)} e^{-\widehat{\Phi}(q)\epsilon} \widehat{W}(\epsilon) \right) \\
& \quad + \frac{q}{q+\theta} \left(1 - \widehat{Z}^{(q), \alpha}(\epsilon) + \frac{e^{-\alpha\epsilon} p_2}{\widehat{\Phi}(q) - \alpha} \widehat{W}^{(q)}(\epsilon) \right) \\
& \quad + \widehat{Z}^{(q+\theta), \alpha}(\epsilon) - \frac{p_3}{\widehat{\Phi}(q+\theta) - \alpha} \widehat{W}^{(q+\theta)}(\epsilon) \\
& \quad \left. - \widehat{Z}^{(\theta+q), \widehat{\Phi}(q)}(\epsilon) + \frac{e^{-\widehat{\Phi}(q)\epsilon} \theta}{\widehat{\Phi}(q+\theta) - \widehat{\Phi}(q)} \widehat{W}^{(q+\theta)}(\epsilon) \right\}, \tag{33}
\end{aligned}$$

where we used the following identities: $\widehat{\Phi}_{\widehat{\Phi}(q)}(\theta) = \widehat{\Phi}(q+\theta) - \widehat{\Phi}(q)$, $\widehat{\Phi}_\alpha(p_2) = \widehat{\Phi}(q) - \alpha$, $\widehat{\Phi}_\alpha(p_3) = \widehat{\Phi}(q+\theta) - \alpha$.

We split the analysis into two cases when the process X is of bounded variation and when $\sigma > 0$. In the first scenario taking $\epsilon = 0$ completes the proof of (20) since $\widehat{W}(0) > 0$. When $\sigma > 0$ then the proof of (21) follows from taking limit $\epsilon \downarrow 0$ and observing that by [12, Thm 3.10] the derivative $\widehat{W}'(0)$ is well-defined. \square

Proof of Proposition 12. The proof is based on strong Markov property and fact that spectrally negative Lévy process creeps upward. Note that:

$$\begin{aligned}
 & \mathbb{E}_x [e^{-\alpha X_{e_q}}, \tau^\theta > e_q] \\
 &= \mathbb{E}_x [e^{-\alpha X_{e_q}}, \tau_0^- > e_q] \\
 &+ \int_0^\infty \mathbb{P}_x(\tau_0^- \leq e_q, -X_{\tau_0^-} \in dz) \\
 &\quad \{ \mathbb{E} [e^{-\alpha X_{e_q}}, \tau_z^+ > e_q] \mathbb{P}(e_\theta > e_q) \\
 &\quad + \mathbb{P}(\tau_z^+ \leq \min(e_\theta, e_q)) \mathbb{E} [e^{-\alpha X_{e_q}}, \tau^\theta > e_q] \}.
 \end{aligned} \tag{34}$$

In (34) we use lack of memory of the exponential distribution.

Then

$$\begin{aligned}
 \mathbb{E}_x [e^{-\alpha X_{e_q}}, \tau^\theta > e_q] &= \mathbb{E}_x [e^{-\alpha X_{e_q}}, \tau_0^- > e_q] \\
 &+ \frac{q}{q+\theta} \int_0^\infty \mathbb{P}_x(\tau_0^- \leq e_q, -X_{\tau_0^-} \in dz) \mathbb{E} [e^{-\alpha X_{e_q}}, \tau_z^+ \geq e_q] \\
 &+ \mathbb{E} [e^{-\alpha X_{e_q}}, \tau^\theta > e_q] \int_0^\infty e^{-\Phi(\theta+q)z} \mathbb{P}_x(\tau_0^- \leq e_q, -X_{\tau_0^-} \in dz).
 \end{aligned}$$

Thus

$$\begin{aligned}
 \mathbb{E}_x [e^{-\alpha X_{e_q}}, \tau^\theta > e_q] &= \mathbb{E}_x [e^{-\alpha X_{e_q}}, \tau_0^- > e_q] \\
 &+ \frac{q}{q+\theta} \int_0^\infty \mathbb{P}_x(\tau_0^- \leq e_q, -X_{\tau_0^-} \in dz) \mathbb{E}_z [e^{\alpha \widehat{X}_{e_q}}, \widehat{\tau}_0^- \geq e_q] \\
 &+ \mathbb{E} [e^{-\alpha X_{e_q}}, \tau^\theta > e_q] \mathbb{E}_x \left[e^{-q\tau_0^- + \Phi(\theta+q)X_{\tau_0^-}}, \tau_0^- < \infty \right].
 \end{aligned}$$

In the next step we use (32) to derive the following expression:

$$\begin{aligned}
\mathbb{E}_x [e^{-\alpha X_{e_q}}, \tau^\theta > e_q] &= \mathbb{E}_x [e^{-\alpha X_{e_q}}, \tau_0^- > e_q] \\
&\quad - \frac{\Phi^2(q)q}{(\Phi(q) - \alpha)(\alpha + \Phi(q))(q + \theta)} \mathbb{E}_x \left[e^{-q\tau_0^- - \alpha X_{\tau_0^-}}, \tau_0^- < \infty \right] \\
&\quad + \frac{\Phi^2(q)q}{(\Phi(q) - \alpha)(\alpha + \Phi(q))(q + \theta)} \mathbb{E}_x \left[e^{-q\tau_0^- + \Phi(q)X_{\tau_0^-}}, \tau_0^- < \infty \right] \\
&\quad + \mathbb{E} [e^{-\alpha X_{e_q}}, \tau^\theta > e_q] \mathbb{E}_x \left[e^{-q\tau_0^- + \Phi(\theta+q)X_{\tau_0^-}}, \tau_0^- < \infty \right].
\end{aligned}$$

By (10) we obtain:

$$\begin{aligned}
\mathbb{E}_x [e^{-\alpha X_{e_q}}, \tau^\theta > e_q] &= \mathbb{E}_x [e^{-\alpha X_{e_q}}, \tau_0^- > e_q] \\
&\quad - \frac{\Phi^2(q)q}{(\Phi^2(q) - \alpha^2)(q + \theta)} \left(e^{-\alpha x} Z^{(q), -\alpha}(x) - \frac{q - \varphi(-\alpha)}{\Phi(q) + \alpha} W^{(q)}(x) \right) \\
&\quad + \frac{\Phi^2(q)qe^{\Phi(q)x}}{(\Phi^2(q) - \alpha^2)(q + \theta)} \\
&\quad + \mathbb{E} [e^{-\alpha X_{e_q}}, \tau^\theta > e_q] \left(e^{\Phi(\theta+q)x} Z^{(q), \Phi(\theta+q)}(x) - \frac{\theta}{\Phi(q + \theta) - \Phi(q)} W^{(q)}(x) \right).
\end{aligned}$$

Finally, for $x \geq 0$:

$$\begin{aligned}
\mathbb{E}_x [e^{-\alpha X_{e_q}}, \tau_0^- > e_q] &= \frac{q}{\Phi(q) + \alpha} W^{(q)}(0) \\
&\quad + \frac{q}{\Phi(q) + \alpha} \left(W^{(q)}(x) - e^{-\alpha x} W^{(q)}(0) - \alpha e^{-\alpha x} \int_0^x e^{\alpha z} W^{(q)}(z) dz \right) \\
&\quad + \frac{\Phi(q)q}{\Phi(q) + \alpha} \left(\alpha e^{-\alpha x} \int_0^x e^{\alpha z} \int_0^z W^{(q)}(y) dy dz - \int_0^x W^{(q)}(y) dy \right)
\end{aligned}$$

which follows from the Lemma 1, integration by parts formula and observation that for $z > 0$:

$$\begin{aligned}
\mathbb{P}(\widehat{X}(e_q) \leq z) &= \mathbb{P}(\widehat{\tau}_z^+ > e_q) \\
&= 1 - \mathbb{E}[e^{-q\widehat{\tau}_z^+}] = \frac{q}{\Phi(q)} W^{(q)}(z) - q \int_0^z W^{(q)}(y) dy
\end{aligned}$$

and hence $\mathbb{P}\left(\widehat{X}(e_q) = 0\right) = \frac{q}{\Phi(q)}W^{(q)}(0)$. This completes the proof of (28). To find $\mathbb{E}\left[e^{-\alpha X_{e_q}}, \tau^\theta > e_q\right]$ we use now the same arguments like in the spectrally positive case introducing stopping time $\tau_\epsilon^\theta = \inf\{t > 0 : t - \sup\{0 \leq s \leq t : X_s \geq \epsilon\} > e_\theta, X_t < 0\}$ which approximates Parisian ruin time and taking $x = \epsilon$ in (28). \square

REFERENCES

- [1] Abate, J. and Whitt, W. (1997) Asymptotics for $M/G/1$ low-priority waiting-time tail probabilities. *Queueing Systems* **25**, 173–233.
- [2] Albrecher, H., Kortschak, D. and Zhoui, D. (2012) Pricing of Parisian options for a jump-diffusion model with two-sided jumps. *Applied Mathematical Finance* **19**(2), 97–129.
- [3] Czarna, I. and Palmowski, Z. (2011) Ruin probability with Parisian delay for a spectrally negative Lévy risk process. *J. Appl. Probab.* **48**(4), 984–1002.
- [4] Chesney, M., Jeanblanc-Picqué, M. and Yor, M. (1997) Brownian excursions and Parisian barrier options. *Adv. in Appl. Probab.* **29**(1), 165–184.
- [5] Dassios, A. and Wu, S. (2009) Ruin probabilities of the Parisian type for small claims. Submitted for publication, see <http://stats.lse.ac.uk/angelos/>.
- [6] Dassios, A. and Wu, S. (2010) Perturbed Brownian motion and its application to Parisian option pricing. *Finance and Stochastics* **14**(3), 473–494.
- [7] G. Doetsch (1974) *Introduction to the Theory and Application of the Laplace Transformation*. Springer, Berlin.
- [8] Haas, B. and Rivero, V. (2012) Quasi-stationary distributions and Yaglom limits of self-similar Markov processes. *Stoch. Proc. Appl.* **122**(12), 4054–4095.
- [9] Henrici, P. (1977) *Applied and Computational Complex Analysis*. vol. 2, Wiley, New York.
- [10] Iglehart, D.L. (1974) Random walks with negative drift conditioned to stay positive. *J. Appl. Probab.* **11**, 742–751.
- [11] Jacka, S.D. and Roberts, G.O. (1995) Weak convergence of conditioned processes on a countable state space. *J. Appl. Probab.* **32**, 902–916.
- [12] Kuznetsov, A., Kyprianou, A. and Rivero, V. (2012) The theory of scale functions for spectrally negative Lévy processes. To appear in *Lévy Matters II*, Lecture Notes in Mathematics, Springer.
- [13] Kyprianou, E.K. (1971) On the quasi-stationary distribution of the virtual waiting time in queues with Poisson arrivals. *J. Appl. Probab.* **8**, 494–507.
- [14] Kyprianou, A.E. (2006) *Introductory Lectures on Fluctuations of Lévy Processes with Applications*. Springer.
- [15] Kyprianou, A. and Palmowski, Z. (2006) Quasi-stationary distributions for Lévy processes. *Bernoulli* **12**(4), 571–581.
- [16] Mandjes, M., Palmowski, Z. and Rolski, T. (2012) Quasi-stationary workload in a Lévy-driven storage system. *Stochastic Models* **28**(3), 413–432.
- [17] Martinez, S. and San Martin, J. (1994) Quasi-stationary distributions for a Brownian motion with drift and associated limit laws. *J. Appl. Probab.* **31**, 911–920.

- [18] Seneta, E. and Vere-Jones, D. (1966) On quasi-stationary distributions in discrete-time Markov chains with a denumerable infinity of states. *J. Appl. Probab.* **3**, 403–434.
- [19] Tweedie, R.L. (1974) Quasi-stationary distributions for Markov chains on a general state space. *J. Appl. Probab.* **11**, 726–741.

MATHEMATICAL INSTITUTE, UNIVERSITY OF WROCLAW, POLAND.

E-mail address: `czarna@math.uni.wroc.pl`

MATHEMATICAL INSTITUTE, UNIVERSITY OF WROCLAW, POLAND.

E-mail address: `zbigniew.palmowski@gmail.com`